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Exterior gauging of an internal supersymmetry and *SU*(2/1) quantum asthenodynamics

(Lie supergroups/Grassmann algebra/Becchi–Rouet–Stora algebra/weak electromagnetic unification/Kalb–Ramond field)

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ABSTRACT **A formally unitary Lagrangian model gauging an internal supersymmetry is proposed. The even subalgebra is gauged as a Yang–Mills theory, while the odd generators are gauged—according to Freedman's method—by skew tensor fields, equivalent dynamically to scalar Higgs fields. Chiral fermions are incorporated by following Townsend's construction and form irreducible supermultiplets graded by their helicity. The application to quantum asthenodynamics is discussed.**

1. This paper presents a model–theory for the gauging of an internal simple supergroup $\mathscr{G}$. Its simple generator superalgebra g is exponentiated with parameters supplied by Grassmann's original exterior algebra Ω of forms over space-time. The corresponding reducible Lie algebra $\tilde{g}$ is given by the even part of the direct product $g \otimes \Omega$—i.e.,

$$\tilde{g} = (g_+ \otimes \Omega_+) \oplus (g_- \otimes \Omega_-). \qquad [1.1]$$

(The $+/-$ indices denote even/odd gradings in both superalgebras.)

We have three aims in presenting these results: (*i*) to present a novel way in which a gauge supergroup may mix internal symmetry with external action over space-time, a point of physical interest; (*ii*) to display the resulting system of generalized connections and curvatures, a mathematical result that might have some applications in differential geometry, and to provide a nontrivial example of a Cartan integrable system; and (*iii*) to suggest an outline for a dynamical realization of a tentative $SU(2/1)$ gauge symmetry describing a constrained asthenodynamic (weak electromagnetic unified) interaction (1–3) and to explain some of the more puzzling features of that model: the group metric, grading by chiralities, and identification of states with ghost statistics in the multiplets.

2. It was recently noted (1–3) that the number of independent arbitrary assumptions required by the algebraically nonsimple $SU(2) \times U(1)$ gauge theory (4, 5) of "unified" weak electromagnetic interactions is greatly reduced by the application of the simple supergroup $SU(2/1) \supset SU(2) \times U(1)$ as a higher constraining internal symmetry. The five independent multiplets $\mathbf{2}(\nu_L^0, e_L^-)$ and $\mathbf{1}(e_R^-)$ selected for the leptons and $\mathbf{2}(u_L^{2/3}, d_L^{-1/3})$, $\mathbf{1}(u_R^{2/3})$, $\mathbf{1}(d_R^{-1/3})$ for the quarks (in any one generation) in the Weinberg–Salam group assignments are replaced (up to statistics) by the two fundamental irreducible representations (3, 6) of $SU(2/1)$: $\mathbf{3}(\nu_L^0, e_L^-/e_R^-)$ and the fractionally charged $\mathbf{4}(u_R^{2/3}/u_L^{2/3},\ d_L^{-1/3}/d_R^{-1/3})$. Moreover, for integer electric charges, the $SU(2/1)$ irrep $\mathbf{4} \rightarrow \mathbf{3} \oplus \mathbf{1}$, so that the $\mathbf{3}$ structure for leptons and the decoupling of ν_R^0 are predicted by $SU(2/1)$. In addition, as against the arbitrary selection of an $SU(2)$ doublet for the spontaneous $SU(2) \times U(1)$ symmetry breakdown in Goldstone–Higgs h^i fields, we find that in $SU(2/1)$, these h^i fit in the adjoint representation, together with the Faddeev–Popov ghosts α^a. At the classification level, thus, $SU(2/1)$ is rather promising; moreover, more recently it has been possible (7) to extend the method to further unification with the strong interactions' conjectured color $SU(3)$. The specific selection of $SU(7/1)$ as the overall simple unifying group (8) produces an anomaly-free $SU(7)$ renormalizable gauge theory with a prediction of eight generations (= 16 flavors), half of them chiral-inverted, thus imposing uniquely the "critical" quantum chromodynamics (i.e., with a factorable Pomeranchuk trajectory). Elsewhere, the pion model also can be superunified (9) by using the superalgebra $Q(3)$.

The main difficulty facing internal supersymmetry is to provide a correct interpretation of the odd generators of the superalgebra. Any irreducible representation is made up of Bose and Fermi particles. Thus, if the supergroup is assumed to commute with the Lorentz group, part of those particles violate the spin statistics theorem. We have assumed (1, 3) that those ghost states should be interpreted as generalized Faddeev–Popov unitarity ghost states, generalizing to every multiplet the pattern of the scalar Higgs field plus Faddeev–Popov ghost sector. In particular, the Yang–Mills $SU(2) \times U(1)$ vector bosons $(W_\mu^\pm, Z_\mu^0, A_\mu^{\mathrm{em}})$ should be completed by a (pair of) isodoublet vector ghost states β_μ^i (and $\bar{\beta}_\mu^i$). In 1981, we constructed a closed irreducible extended Becchi–Rouet–Stora (BRS) algebra (10) for just those fields, however without being able to explain the role of the $\beta_\mu^i(\bar{\beta}_\mu^i)$ in unitarity. In the new framework presented in this paper, the BRS algebra will be modified, and the $\beta_\mu^i(\bar{\beta}_\mu^i)$ will appear naturally as the vector ghosts of a Bose skew tensor field $B_{\mu\nu}^i$, gauging g_-. In a way, we shall explicitly construct a realization of the operator ε_μ conjectured by us earlier (3). The price paid is that our new Bose algebra $\tilde{g}$ is reducible with $SU(2) \times U(1)$ as its maximal simple Lie subalgebra. However, two other difficulties of the model are resolved at the same time. The algebra is a Lie algebra and, therefore, should be normalized by traces rather than by supertraces, thus yielding a positive definite metric in the $SU(2) \times U(1)$ sector and confirming a Weinberg angle of $\sin^2\theta = \frac{1}{4}$. The reducibility of the boson vector algebra is necessary because it implies the vanishing of the Killing metric in the Higgs sector, therefore allowing the use of the $SU(2) \times U(1)$ positive norm in the Higgs and matter sectors.

The second difficulty resolved by the present approach is the detailed understanding of the role of the matter ghost fields and of the grading of the physical spinor fields by their helicity. These results are explained in section **6**.

As yet we have no complete understanding of the symmetry

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Abbreviation: BRS, Becchi–Rouet–Stora.

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breaking and cannot confirm our 250-GeV conjecture (3) for the mass of the physical Higgs field.

3. In order to exponentiate a superalgebra $g = g^+ + g^-$, it is always necessary to introduce an exterior algebra of anticommuting parameters. In this way one constructs a Lie group $\mathcal{G}(g,\Omega) = \exp(\Omega^+ \otimes g^+ + \Omega^- \otimes g^-)$. This group always admits an underlying Lie algebra $\tilde{g}$, generally reducible, whose dimensionality depends on Ω,

$$\tilde{g} = \Omega^+ \otimes g^+ \oplus \Omega^- \otimes g^-$$

$$\dim(\tilde{g}) = \tfrac{1}{2}\dim(g)\cdot\dim(\Omega). \qquad [3.1]$$

However, some real forms of simple supergroups do not admit an underlying superalgebra (11). In the following, we shall work out a model where Ω is taken to be the exterior algebra of forms over space-time itself. Therefore, the corresponding gauge theory is highly soldered to the base space. Nevertheless, most of the attractive properties of differential geometry are maintained, and our construction provides a nontrivial example of a Cartan integrable system (12). The generators of $\tilde{g}$ will be denoted as $\lambda_a = \mu_a$, $\lambda_i^\mu = \mu_i\, dx^\mu$, $\lambda_a^{\mu\nu} = \mu_a\, dx^\mu \wedge dx^\nu, \ldots$ ($\mu_a \in g^+$, $\mu_i \in g^-$).

The Lie bracket is defined as the exterior product for the forms times the Lie superbracket $[\mu_M, \mu_N\}$. For instance

$$[\lambda_i^\mu, \lambda_j^\nu\} = dx^\mu \wedge dx^\nu \{\mu_i, \mu_j\} = f_{ij\rho\sigma}^{\mu\nu a}\, \lambda_a^{\rho\sigma}. \qquad [3.2]$$

The Jacobi identity is automatically satisfied. Throughout this paper we use square brackets to denote $SU(2) \times U(1)$ commutation relationships and curly brackets to denote the relationships specific to the superalgebra $SU(2/1)$. They really denote commutators and anticommutators of number matrices, once all exterior products have been evaluated.

Such a generalized gauge theory involves a generalized system of connections, skew-symmetric contravariant, and Bose tensor gauge fields A_μ^a, $B_{\mu\nu}^i$, $C_{\mu\nu\rho}^a$, and $E_{\mu\nu\rho\sigma}^i$ of alternating supergroup gradings, saturating the dimensionality of space-time forms. Under an infinitesimal transformation with parameter $\tilde{\varepsilon}(\varepsilon^a, \varepsilon_\mu^i, \varepsilon_{\mu\nu}^a, \varepsilon_{\mu\nu\rho}^i, \varepsilon_{\mu\nu\rho\sigma}^a)$, the gauge fields vary according to

$$\begin{aligned}
\delta A &= -D\varepsilon^{(0)} := -d\varepsilon^{(0)} - [A,\varepsilon^{(0)}] &&=: \tilde{D}\varepsilon^{(0)}\\
\delta B &= -D\varepsilon^{(1)} - [B,\varepsilon^{(0)}] &&=: \tilde{D}\varepsilon^{(1)}\\
\delta C &= -D\varepsilon^{(2)} - \{B,\varepsilon^{(1)}\} - [C,\varepsilon^{(0)}] &&=: \tilde{D}\varepsilon^{(2)}\\
\delta E &= -D\varepsilon^{(3)} - [B,\varepsilon^{(2)}] - [C,\varepsilon^{(1)}] - [E,\varepsilon^{(2)}] &&=: \tilde{D}\varepsilon^{(3)},
\end{aligned} \qquad [3.3]$$

where D denotes the λ_a covariant differential with gauge field A_μ^a, d is the external differential, $A = A_\mu^a\, \mu_a\, dx^\mu$, $B = \tfrac{1}{2} B_{\mu\nu}^i\, \mu_i\, dx^\mu \wedge dx^\nu$, $C = \tfrac{1}{6} C_{\mu\nu\rho}^a\, dx^\mu \wedge dx^\nu \wedge dx^\rho$, $E = \tfrac{1}{24} E_{\mu\nu\rho\sigma}^i\, \mu_i\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$, and exterior products are implied.

These equations define the action of the generalized covariant derivative $\tilde{D}$. The generalized curvature $\tilde{F}$ is similarly defined,

$$\begin{aligned}
\tilde{F} &= (F^a, G^i, H^a)\\
F^a &= dA + \tfrac{1}{2}[A,A]^a\\
G^i &= (DB)^i\\
H^a &= (DC)^a + \tfrac{1}{2}\{B,B\}^a.
\end{aligned} \qquad [3.4]$$

These curvatures transform covariantly,

$$\begin{aligned}
\delta\tilde{F} = [\tilde{\varepsilon},\tilde{F}]:\ \delta F &= -[F,\varepsilon^{(0)}]\\
\delta G &= -[G,\varepsilon^{(0)}] - [F,\varepsilon^{(1)}]\\
\delta H &= -[H,\varepsilon^{(0)}] - \{G,\varepsilon^{(1)}\} - [F,\varepsilon^{(2)}],
\end{aligned} \qquad [3.5]$$

and satisfy the Bianchi identity

$$\begin{aligned}
\tilde{D}\tilde{F} = 0:\ DF &= 0\\
DG + [B,F] &= 0.
\end{aligned} \qquad [3.6]$$

4. Along the same lines, an irreducible representation $R = R^+ + R^-$ of the superalgebra g will give rise to a representation $\tilde{R}$ of $\tilde{g}$. Denote by $\tilde{\phi}$ a system of 0,1,2... forms taking their values alternatively in R^+ and R^-, $\tilde{\phi}(\phi^A, \psi_\mu^I\, dx^\mu, \Xi_{\mu\nu}^A\, dx^\mu \wedge dx^\nu, \ldots)$.

The representation is defined by the transformation rules,

$$\begin{aligned}
\delta\phi &= [\varepsilon^{(0)}, \phi]\\
\delta\psi &= [\varepsilon^{(0)}, \psi] + \{\varepsilon^{(1)}, \phi\}\\
\delta\Xi &= [\varepsilon^{(0)}, \Xi] + \{\varepsilon^{(1)}, \psi\} + [\varepsilon^{(2)}, \phi],
\end{aligned} \qquad [4.1]$$

with the Jacobi identity automatically satisfied. In the $SU(2/1)$ system, the connections (Eqs. **3.3**) are the generalized gauge fields and the curvatures (Eqs. **3.4**) are their field strengths. For the leptons (and quarks) we have a doublet left-spinor $(\phi)_L$ together with a singlet (two singlets) left-vector-spinor $(\psi_\mu)_L$, etc., all with Fermi statistics.

5. We now show a free-field Lagrangian such that it has precisely the same physical degrees of freedom as the Weinberg–Salam model. Given a skew p-tensor $\phi_{\mu\nu\ldots}$ and its exterior derivative (or generalized curl), this is

$$\mathcal{L}_p = -\tfrac{1}{2}p!\,(\partial_{[\mu}\, \phi_{\nu\rho\ldots]})^2. \qquad [5.1]$$

For a scalar ϕ, this is the Klein–Gordon equation; for ϕ_μ, this is Maxwell's Lagrangian; for $\phi_{\mu\nu}$ this is Kalb–Ramond field (13, 14). In N dimensions (N-1 space-type dimensions and one time dimension) and for the massless Lagrangian (Eq. **5.1**), there is a duality equivalence (15) between p and N-p-2 forms (this is Hodge duality in the transverse dimensions).

It can be shown that the number of physical degrees of freedom n for the gauge fields (Eqs. **3.3**) is precisely given by adding up the number $\binom{N}{k}$ of components of an antisymmetric k-indices tensor in N dimensions, together with the number of dy-contracted components of its complexified (10, 16) geometrical vertical complements [in the direction of the fiber y^M, in the bundle manifold (9, 17)], ghosts counting negatively. For the forms in Eqs. **3.3**, with the fiber-complexified forms denoted by a caret,

$$\hat{A}^a = A_\mu^a\, dx^\mu + A_M^a\, dy^M + A_{\bar{N}}^a\, d\bar{y}^{\bar{N}} = A_\mu^a\, dx^\mu + \alpha^a + \bar{\alpha}^a$$

$$\begin{aligned}
\hat{B}^i &= \tfrac{1}{2} B_{\mu\nu}^i\, dx^\mu \wedge dx^\nu + B_{\mu M}^i\, dx^\mu \wedge dy^M + B_{\mu\bar{N}}^i\, dx^\mu \wedge d\bar{y}^{\bar{N}}\\
&\quad + \tfrac{1}{2} B_{MN}^i\, dy^M \wedge dy^N + B_{M\bar{N}}^i\, dy^M \wedge d\bar{y}^{\bar{N}}\\
&\quad + \tfrac{1}{2} B_{\bar{M}\bar{N}}^i\, dy^{\bar{M}} \wedge dy^{\bar{N}}\\
&= \tfrac{1}{2} B_{\mu\nu}^i\, dx^\mu \wedge dx^\nu + B_\mu^i\, dx^\mu\\
&\quad + \bar{\beta}_\mu^i\, dx^\mu + b^i + h^i + \bar{b}^i
\end{aligned}$$

$$\begin{aligned}
\hat{C}^a &= \tfrac{1}{6} C_{\mu\nu\rho}^a\, dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{2} C_{\mu\nu M}^a\, dx^\mu \wedge dx^\nu \wedge dy^M\\
&\quad + \tfrac{1}{2} C_{\mu\nu\bar{N}}^a\, dx^\mu \wedge dx^\nu \wedge d\bar{y}^{\bar{N}}\\
&\quad + \tfrac{1}{2} C_{\mu MN}^a\, dx^\mu \wedge dy^M \wedge dy^N\\
&\quad + C_{\mu M\bar{N}}^a\, dx^\mu \wedge dy^M \wedge d\bar{y}^{\bar{N}} + \tfrac{1}{2} C_{\mu\bar{M}\bar{N}}^a\, dx^\mu \wedge dy^{\bar{M}} \wedge d\bar{y}^{\bar{N}}
\end{aligned}$$

$$+ \tfrac{1}{6} C^a_{MNP}\, dy^M \wedge dy^N \wedge dy^P$$
$$+ \tfrac{1}{2} C^a_{MN\bar{P}}\, dy^M \wedge dy^N \wedge d\bar{y}^{\bar{P}}$$
$$+ \tfrac{1}{2} C^a_{M\bar{N}\bar{P}}\, dy^M \wedge d\bar{y}^{\bar{N}} \wedge d\bar{y}^{\bar{P}}$$
$$+ \tfrac{1}{6} C^a_{\bar{M}\bar{N}\bar{P}}\, d\bar{y}^{\bar{M}} \wedge d\bar{y}^{\bar{N}} \wedge d\bar{y}^{\bar{P}}$$
$$= \tfrac{1}{6} C^a_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{2} (\Gamma^a_{\mu\nu} + \bar{\Gamma}^a_{\mu\nu})\, dx^\mu \wedge dx^\nu$$
$$+ (c^{a+}_\mu + c^a_\mu + c^{a-}_\mu) dx^\mu + \gamma^{a---}$$
$$+ \gamma^{a+} + \gamma^{a-} + \gamma^{a+++}, \qquad \text{etc.} \qquad [5.2]$$

Latin letters denote Bose fields; Greek, Fermi ghost fields. We have (per internal index) for $N = 4$,

$$n(A) = \binom{4}{1} - (2 \times 1) = 2$$
$$n(B) = \binom{4}{2} - (2 \times 4) + 3 = 1$$
$$n(C) = \binom{4}{3} - (2 \times 6) + (3 \times 4) - 4 = 0$$
$$n(E) = \binom{4}{4} - (2 \times 4) + (3 \times 6) - (4 \times 4) + 5 = 0. \qquad [5.3]$$

Actually, the $(y,\bar{y})$ could be replaced by anticommuting $\theta,\bar{\theta}$ parameters (18) without altering in any way our counting procedure. We note that $\hat{B}^i$ contains a scalar real h^i multiplet, required in the $SU(2/1)$ irreps $8'(\alpha^a,h^i)$ or $\bar{8}'(\bar{\alpha}^a,h^i)$, in the Curci–Ferrari type of symmetric-complexified algebra of ghosts (16). The higher forms C and E do not contribute to the physical spectrum, nor would the total contribution of the system of nonvanishing ghosts of a higher tensor.

6. Taking the lepton triplet of $SU(2/1)$ as an example, we use the Weyl action for the massless left isodoublet $\phi\,(\nu^0_L, e^-_L)$. On the other hand, we use the Townsend (19, 20) action for ψ^L_μ, an isosinglet left vector-spinor. We denote by $\Upsilon_{\mu\nu}$ an auxiliary Dirac spinor two-form and by ψ^R_μ an auxiliary right spinor one-form (we use 2-spinor notation):

$$\mathcal{L}_R = \varepsilon_{\lambda\mu\nu\rho} (\overline{\Upsilon^L_{\lambda\mu}} (\partial_\nu \psi^L_\rho + \sigma_\nu \psi^R_\rho) + \overline{\psi^L_\lambda} \sigma_\mu \partial_\nu \psi^L_\rho + \overline{\Upsilon^R_{\lambda\mu}} \partial_\nu \psi^R_\rho). \qquad [6.1]$$

The $\Upsilon_{\lambda\mu}$ equations of motion enforce the constraints

$$d\psi^L + \sigma \wedge \psi^R = 0,\ d\psi^R = 0, \qquad [6.2]$$

which have the solution (20)

$$\psi^L = d\tau^L + \sigma \wedge \nu^R$$
$$\psi^R = d\nu^R, \qquad [6.3]$$

and the Lagrangian is equivalent up to the equations of motions to a Weyl Lagrangian, in terms of the right spinor ν^R

$$\mathcal{L}'_R \overset{\circ}{=} \varepsilon_{\lambda\mu\nu\rho}\, \overline{\nu^R} \wedge \sigma_\lambda \wedge \sigma_\mu \wedge \sigma_\nu \wedge \partial_\rho\, \nu^R = \overline{\nu^R}\, \sigma^\mu\, \partial_\mu\, \nu^R. \qquad [6.4]$$

The left vector-spinor ψ^L_μ in Eq. **6.1** thus is seen to represent physically a right spinor ν^R, thus fitting the right isosinglet fermions such as e^-_R. On the other hand, the formal left vector-spinor ψ^L_μ is a one-form whose vertical complement is given by

$$\hat{\psi}^L = \psi^L_\mu\, dx^\mu + \psi^L_M\, dy^M + \psi^L_{\bar{N}}\, d\bar{y}^{\bar{N}}$$
$$= \psi^L_\mu\, dx^\mu + \chi^{L+} + \chi^{L-}. \qquad [6.5]$$

The bosonic vector-spinor ghosts χ^{L+} (and χ^{L-}) can now be identified with the ghost state with ψ^R internal isoscalar quantum numbers (1, 3, 9) appearing together with the ϕ^L doublets in 3 or 4 of $SU(2/1)$ [and its symmetric Curci–Ferrari extension for χ^{L-} (10)].

It is indeed remarkable that the Townsend Lagrangian thus should explain both the ghost statistics and the chiral inversion in the $SU(2/1)$ matter multiplets, explaining another puzzling feature brought out by the classifying supergroup.

7. The Interacting Lagrangian. For several reasons, there is no trivial generalization of the Abelian Lagrangian to the non-Abelian case. On one hand, the Lie algebra is reducible, and its Killing metric is nonzero only in the A^a_μ sector, so only the Yang–Mills vector Lagrangian comes out as a natural invariant. On the other hand, if we consider as a Lagrangian for the $B^i_{\mu\nu}$ the term

$$\mathcal{L}_B = -\tfrac{1}{12} (G^i_{\mu\nu\rho})^2, \qquad [7.1]$$

using the $SU(2) \times U(1)$ symmetric δ_{ij} metric, the $B^i_{\mu\nu}$ equation of the motion $D_\mu G_{\mu\nu\rho} = 0$ enforces the constraint

$$D_\mu D_\nu\, G_{\mu\nu\rho} = [F_{\mu\nu}, G_{\mu\nu\rho}] = 0. \qquad [7.2]$$

The natural invariance of the Lagrangian under covariant BRS variations $SB_{\mu\nu} = D_{[\mu}\, \beta_{\nu]}$ is also lost if the constraint is not satisfied:

$$S\mathcal{L}_B = G_{\mu\nu\rho}\, D_\mu D_\nu \beta_\rho = [G_{\mu\nu\rho}, F_{\mu\nu}] \beta_\rho, \qquad [7.3]$$

where β_ν is the vector ghost in Eqs. **5.2** and $S = s + [\alpha,\ ]$ of ref. 21. Exactly the same defects plague the vector-ghost sector of the once gauge-fixed Freedman–Townsend Lagrangian $\mathcal{L}_\beta = D_{[\mu} \bar{\beta}_{\nu]}\, D_\mu\, \beta_\nu$ in the covariant quantization formalism (refs. 21 and 22). In the latter case, we have recently proposed, in collaboration with Laurent Baulieu, a solution based on the existence of a secondary gauge invariance of the classical Lagrangian under a transformation with scalar ghost parameter κ, $SB = [F,\kappa] = DD\kappa$.

The usual BRS quantization procedure (23, 24) then leads naturally to a modified constraint-free and gauge-invariant Lagrangian for the β' field:

$$\mathcal{L}'_\beta = S\bar{S}\, (B_{\mu\nu})^2 = (D_\mu\, (\bar{\beta}_\nu + D_\nu \bar{\kappa}))(D_\mu (\beta_\nu + D_\nu \kappa)) + \dots \qquad [7.4]$$

In the present framework, the constraints manifest themselves already at the classical level. The following method, however, permits a direct transition from $\mathcal{L}_\beta$ to $\mathcal{L}'_\beta$ considered as classical Lagrangians. Inspired by Dirac's work, we simply introduce a Lagrange multiplier K_μ, whose equation of the motion enforces the differential constraints and consider the Lagrangian

$$\mathcal{L}'_B = -\tfrac{1}{12} (\varepsilon^{\mu\nu\rho\sigma} (D_\mu (B_{\nu\rho} + D_\nu K_\rho)))^2. \qquad [7.5]$$

The equations of motion are

$$D_\mu (D_{[\mu} (B_{\nu\rho]} + D_{[\nu} K_{\rho]})) = 0$$
$$[F_{\mu\nu}, D_{[\mu}\, B_{\nu\rho]}] = 0, \qquad [7.6]$$

and the system is now closed. At the same time, $\mathcal{L}'_\beta$ is invariant under the nilpotent BRS algebra (refs. 17 and 22) involving fields and ghosts from Eqs. **5.2**, and with κ as the ghost of K_μ,

$$S\alpha = \tfrac{1}{2} [\alpha,\alpha]$$
$$SA_\mu = \partial_\mu \alpha$$

$$\begin{aligned} Sb &= 0 \\ S\kappa &= -b \\ S\beta_\mu &= D_\mu b \\ SK_\mu &= -(\beta_\mu + D_\mu \kappa) := -\beta'_\mu \\ SB_{\mu\nu} &= D_{[\mu}\,\beta_{\nu]} + [F_{\mu\nu},\kappa] = D_{[\mu}\,\beta'_{\nu]}. \end{aligned} \qquad [7.7]$$

We have explicitly recovered our solution of Townsend's problem as a subcase. Incidentally, the possibility of extending the BRS algebra of Townsend's σ model by the inclusion of K_μ indicates that Eq. **7.5** might be an admissible counterterm in that theory. The reciprocal is not true. Here, all the fundamental fields—i.e., those which appear in $\tilde{B}(B_{\mu\nu},\beta_\mu,b)$—have canonical dimension one. The K_μ and its pair of ghosts have dimension zero. In a proper gauge they are expected to decouple, therefore ensuring *(i)* the renormalizability of the theory by power counting and *(ii)* its formal unitarity.

Our method admits a direct generalization to the $C^a_{\mu\nu\rho}$ field, whereas the $E^i_{\mu\nu\rho\sigma}$ has no curl in four dimensions. The complete classical gauge Lagrangian can be written involving the auxiliary two-form $\tilde{L}(L_{\mu\nu},\lambda_\mu,l)$

$$\mathcal{L} = -\tfrac{1}{4}\,(F_{\mu\nu}(A))^2 - \tfrac{1}{12}\,(D_{[\mu}\,B'_{\nu\rho]})^2$$

$$-\tfrac{1}{48}\,(D_{[\mu}\,C'_{\nu\rho\sigma]} + \tfrac{1}{2}\,\{B'_{\mu\nu},B'_{\rho\sigma}\})^2$$

$$B'_{\mu\nu} := B_{\mu\nu} + D_{[\mu}\,K_{\nu]},\; C'_{\mu\nu\rho} := C_{\mu\nu\rho} + D_{[\mu}\,L_{\nu\rho]}. \qquad [7.8]$$

The squares are computed using the δ_{ab}, δ_{ij} metric, implying $\sin^2\theta_\omega = 0.25$. This Lagrangian is invariant under the nilpotent BRS algebra given above augmented by the $C_{\mu\nu\rho}$ sector:

$$S\gamma = \tfrac{1}{2}\,\{b,b\},\; Sl = -\gamma - \tfrac{1}{2}\,\{\kappa,b\} = -\gamma'$$

$$Sc_\mu = D_\mu\,\gamma' + \{\beta',b\},$$

$$S\lambda_\mu = -c_\mu - D_\mu l - \{K_\mu,b\} := -c'_\mu$$

$$S\Gamma_{\mu\nu} = D_{[\mu}\,c'_{\mu]} + \tfrac{1}{2}\,\{\beta'_{[\mu},\beta'_{\nu]}\} + \{B'_{\mu\nu},b\}$$

$$SL_{\mu\nu} = -\Gamma_{\mu\nu} - D_{[\mu}\,\lambda_{\nu]} - \{B'_{\mu\nu},\kappa\} - \tfrac{1}{2}\,\{\beta'_{[\mu},K_{\nu]}\} := -\,\Gamma'_{\mu\nu}$$

$$SC_{\mu\nu\rho} = D_{[\mu}\,\Gamma'_{\nu\rho]} + \{B'_{[\mu\nu},\beta'_{\rho]}\}. \qquad [7.9]$$

At the linearized level, we recover the Weinberg–Salam spectrum.

Note that the closure of the BRS algebra is equivalent, through complexification of the y and projection in the dy, $d\bar{y}$ sectors, to the closure of the extended Curci–Ferrari algebra. No additional calculations are required.

8. Let us now construct a BRS algebra (17, 22) for chiral spinor fields. We start from a left chiral spin 1/2 R^+ multiplet ϕ^L and write:

$$S\phi^L = 0. \qquad [8.1]$$

We now introduce a left chiral vector spinor ψ^L_μ, its left chiral Bose spin 1/2 ghost x^{L+}, and an auxiliary spin 1/2 fermion η^L, all valued in R^-. The nilpotent BRS algebra is uniquely defined as

$$Sx^+ = -\{b,\phi\}$$

$$S\eta = x^+ + \{\kappa,\phi\}$$

$$S\psi_\mu = D_\mu x^+ + \{\beta_\mu,\phi\} + \{\kappa,D_\mu\phi\}. \qquad [8.2]$$

We can now construct the covariant differential

$$\tilde{D}_{[\mu}\,\psi_{\nu]} = D_{[\mu}\,\psi_{\nu]} + \{B_{\mu\nu},\phi\} + \{K_{[\mu},D_{\nu]}\phi\} + [F_{\mu\nu},\eta]$$

$$S\tilde{D}\psi = 0, \qquad [8.3]$$

and the invariant spinor

$$\tilde{\psi}_\mu = \psi_\mu + \{K_\mu,\phi\} + D_\mu\,\eta$$

$$S\tilde{\psi}_\mu = 0. \qquad [8.4]$$

Townsend's Lagrangian now generalizes into:

$$\begin{aligned} \mathcal{L}_{1/2} = \varepsilon^{\lambda\mu\nu\rho}\,(&\overline{\Upsilon^L_{\lambda\mu}}\,(\tilde{D}_\nu\,\psi^L_\rho + \sigma_\nu\,\tilde{\psi}^R_\rho) \\ &+ \overline{\Upsilon^R_{\lambda\mu}}\,\tilde{D}_\nu\,\psi^R_\rho + \overline{\psi^L_\lambda}\,\sigma_\mu\,\tilde{D}_\nu\,\psi^L_\rho) \\ &+ \overline{\phi^L}\,\not{D}\,\phi^L. \end{aligned} \qquad [8.5]$$

At the linearized level, we recover Townsend's Lagrangian and propagate a left doublet (ν^0_L, e^-_L) and a right singlet (e^-_R). The dynamical part of the Lagrangian is the generalized Rarita–Schwinger term containing a coupling $\overline{\psi^L_\lambda}\,\sigma_\mu\,\{B_{\nu\rho},\phi\}$, which replaces the usual mass term of the Weinberg–Salam model.

9. The pattern of spontaneous symmetry breakdown is crucial for the application of the model to experiment. Obviously, the tensor $B^i_{\mu\nu}$ may not take a vacuum expectation value without breaking at the same time the Lorentz group. However, its central scalar Bose ghost $h^i = B^i_{M\bar{N}}\,dy^M \wedge d\bar{y}^N$ has the quantum numbers of the ordinary Higgs. Furthermore the term $(DC)^2$ contains $(B^i_{\mu\nu})^4$, which generalizes in the ghost expansion to $(h^i)^4$. Therefore, we conjecture that, in the fully quantized theory, $\langle h^i\rangle_0 \neq 0$. We remark that a term $S\bar{S}(\bar{\psi}\cdot\sigma\eta) = \bar{\psi}\cdot\sigma\,\{h,\phi\} + \ldots$ is BRS admissible and represents an arbitrary (electron) mass term. Spontaneous symmetry breakdown also seems intertwined deeply with the problem of the differential constraints in this theory. Indeed, the theory is unstable for $g \to 0$ as the constraint $[F,{}^*G] \approx [dA,{}^*dB] = 0$ is present for $g \neq 0$ but is of order 0 in g. The constraint obviously admits as a solution the usual symmetry breaking pattern, G^i in the group direction μ_6 and F^a confined to the photon direction $1/\sqrt{2}\,(\mu_3 + \sqrt{3}\,\mu_8)$.

10. In conclusion, using the dx^μ generators of the original Grassmann exterior algebra over space-time, we have associated to any simple superalgebra a reducible Lie algebra whose connection defines a set of skew tensor fields. The Bianchi identity is maintained and defines a Cartan integrable system. The corresponding gauge theory is BRS invariant, seems formally unitary and renormalizable provided a set of dimension 0 auxiliary fields, which play the role of Dirac multipliers of the differential constraints, can be consistently eliminated. Applied to $SU(2/1)$ quantum asthenodynamics, the model successfully restricts the arbitrariness of the Weinberg–Salam model. It yields a positive definite physical subspace and a consistent interpretation of all ghost states and explains the grading of the quark and lepton multiplets by their chirality.

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